Due Fri

8.1 – General Linear Transformations

We saw $T: R^n \to R^m$

Now: $T: M_{23} \to P_5$

Definition: (analogous to Theorem 1.8.2)
If $T : V \to W$ is a mapping from a vector space V to a vector space W, then T is called a **linear transformation** from V to W if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars k:
a) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)
b) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V.

Theorem 8.1.1 (analogous to Theorem 1.8.1)
If $T : V \to W$ is a linear transformation, then:
a) $T(\mathbf{0}) = \mathbf{0}$.
b) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.
c) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

What if $T(\vec{0}) \neq \vec{0}$?
T is not linear.

pf (a): Let $T: V \to W$ be a linear transformation from vector space V to vector space W. $T(k\vec{v}) = kT(\vec{v})$ $\forall \vec{v} \in V$. Let $k = 0$. Then for $\vec{v} \in V$,

$$T(0) = T(0\vec{v}) = 0\,T(\vec{v}) = \vec{0}.$$

Examples of Linear transformations

- $T : V \to W$ defined by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (the zero transformation)
- $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ (the identity operator)
- $T : V \to V$ defined by $T(\mathbf{x}) = c\mathbf{x}$ (contraction if $0 < c < 1$ and dilation if $c > 1$)
- $T : V \to R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ (inner product of $\mathbf{x}$ with $\mathbf{x}_0$) [We'll see this in Ch. 6]
- $T : M_{nn} \to M_{nn}$ defined by $T(A) = A^T$

x	$f(x)$
a	$f(a)$
b	$f(b)$
c	$f(c)$
d	$f(d)$

- $T : V \to R^n$ defined by $T(f) = (f(x_1),\ f(x_2),\ \ldots, f(x_n))$, where V is a subspace of $F(-\infty,\ \infty)$, and $x_1, x_2, \ldots, x_n$ is a sequence of real numbers (evaluation transformation)
- $D : C^1(-\infty, \infty) \to F(-\infty,\ \infty)$ defined by $D(\mathbf{f}) = f'(x)$ (differentiation)
- $J : C(-\infty, \infty) \to C^1(-\infty, \infty)$ defined by $J(\mathbf{f}) = \int_0^x f(t)\, dt$ (integration)

Theorem 8.1.2 Let $T : V \to W$ be a linear transformation, for which the vector space V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v}$ in V can be expressed as
$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \ldots + c_n T(\mathbf{v}_n)$ where $c_1, c_2, \ldots,\ c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination of the vectors in the basis S.

Pf: Let T be as described and $\vec{v} \in V$. Since S is a basis, $\exists\, c_i \ni \vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i$.

$T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} T(c_i \vec{v}_i) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$

$= c_i \sum_{i=1}^{n} T(\vec{v}_i)$. ✓ (T is linear)

#20 Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2,\ 1)$ and $\mathbf{v}_2 = (1,\ 3)$, and let $T : R^2 \to R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1,\ 2,\ 0)$ and $T(\mathbf{v}_2) = (0,\ -3,\ 5)$. Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2,\ -3)$.

When we did this in 1.8, we wanted to find the standard matrix for the transformation. Here, we will find the formula.

We start by expressing $\vec{x} = (x_1, x_2)$ as a linear combination of $\vec{v}_1$ & $\vec{v}_2$:

$$(x_1, x_2) = c_1(-2,1) + c_2(1,3)$$

$$\left[\begin{array}{cc|c} -2 & 1 & x_1 \\ 1 & 3 & x_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3/7\, x_1 + 1/7\, x_2 \\ 0 & 1 & 1/7\, x_1 + 2/7\, x_2 \end{array}\right]$$

$$T(\vec{x}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$$

$$T(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1,2,0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0,-3,5)$$

$$T(x_1, x_2) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{6}{7}x_1 + \frac{2}{7}x_2, 0\right) + \left(0, -\frac{3}{7}x_1 - \frac{6}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$$

$$\boxed{T(x_1, x_2) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)}$$

$$T(2,-3) = \left(\frac{6}{7} + \frac{3}{7}, -\frac{18}{7} + \frac{12}{7}, \frac{10}{7} - \frac{30}{7}\right)$$

$$\boxed{T(2,-3) = \left(\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7}\right)}$$

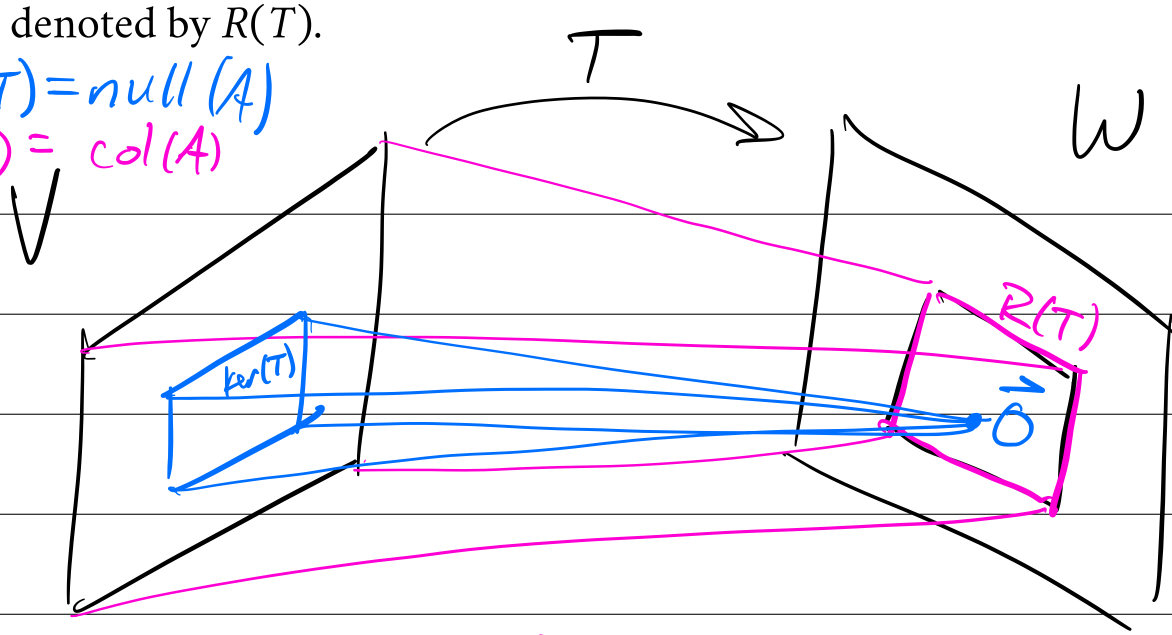
Definition: (analogous to definitions seen in Section 4.2 (**kernel**) and Section 1.8 (**range**); also related to Definition 2 of Section 4.8 (**null space**) and, by Theorem 4.8.1, column space)

domain

If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

$\ker(T) = \text{null}(A)$

$R(T) = \text{col}(A)$



If $T_A : V \to \vec{w}$ is $A(\vec{x})$, then $R(T) = Col(A)$

#6 Determine whether the mapping T is a linear transformation, and if so, find its kernel.

$T : M_{22} \to R$, where

a. $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

b. $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$

b) No $T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = (a_1+a_2)^2 + (b_1+b_2)^2$

$= a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2 \neq a_1^2 + b^2$

$= T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

OR: $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = k^2a^2 + k^2b^2 = k^2(a^2+b^2) \neq kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

a) streamlined:

$$T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ka_1 + a_2 & kb_1 + b_2 \\ kc_1 + c_2 & kd_1 + d_2 \end{bmatrix}\right)$$

$$= 3(ka_1 + a_2) - 4(kb_1 + b_2) + (kc_1 + c_2) - (kd_1 + d_2)$$

$$= k(3a_1 - 4b_1 + c_1 - d_1) + (3a_2 - 4b_2 + c_2 - d_2)$$

$$= kT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_1 & d_1 \end{bmatrix}\right)$$

T is linear

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, 3a + 4b + c - d = 0 \right\}$$

#10 Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

a. x^2

b. 0

c. $1 + x$

d. $-x$

$\{ p \in P_2 \mid xp(x) = 0 \}$

domain

range

These are polynomials of degree less than or equal to 2 such that multiplying by x gives 0.

#11 Let $T : P_2 \to P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

a. $x + x^2$

b. $1 + x$

c. $3 - x^2$

d. $-x$

$T(p(x)) = x\,p(x)$

$R(T)$ is polynomials of degree ≤ 3 with constant term 0.

#25 Let $T_A : R^4 \to R^3$ be multiplication by A. Find a basis for the kernel of T_A, and then find a basis for the range of T_A that consists of column vectors of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

rref of A is $\begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad x_1 = \frac{10}{7}x_4$, $x_2 = \frac{2}{7}x_4$, $x_3 = 0$

Let $x_4 = 7t$. Then $\vec{x} = \begin{bmatrix} 10t \\ 2t \\ 0 \\ 7t \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} t$

Basis for kernel: $\left\{ \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$.

Pivots are in columns 1, 2, 3, so a basis for $R(T)$ is $\left\{ \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

Theorem 8.1.3 If $T : V \to W$ is a linear transformation, then:
a) The kernel of T is a subspace of V.
b) The range of T is a subspace of W.

Definition: (analogous to Definition 1 of Section 4.9)
Let $T : V \to W$ be a linear transformation. In the case that the range of T is finite-dimensional its dimension is called the **rank of T**, denoted by rank(T); and if the kernel of T is finite-dimensional, then its dimension is called the **nullity of T**, denoted by nullity(T).

Theorem 8.1.4 Dimension Theorem for Linear Transformations
(generalization of Theorem 4.9.2)
If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and
$\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

#13 In each part, use the given information to find the nullity of the linear transformation T.

dim 5 dim 6

a. $T : R^5 \to P_5$ has rank 3.
b. $T : P_4 \to P_3$ has rank 1.
c. The range of $T : M_{mn} \to R^3$ is R^3.
d. $T : M_{22} \to M_{22}$ has rank 3.

a) $5 - 3 = \boxed{2 = \text{nullity}(T)}$

b) $5 - 1 = \boxed{4 = \text{nullity}(T)}$

c) $\boxed{mn - 3 = \text{nullity}(T)}$

d) $4 - 3 = \boxed{1 = \text{nullity}(T)}$